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# An Extension of the Theory of Oscillating Cup Viscometers

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The theory of the fluid motion in the interior of an oscillating or rotating cup is reexamined. The quantity of interest in viscometry is the torque exerted by the fluis on the sides and rims of the cup. In this paper expressions for the torque are obtained for geometries for which the cup height approaches a fluid boundary layer thickness. Interest in such geometries is due to viscosity measurements made in mixtures in the critical region where cups of small height are used in order to minimize gravity effects.

**KEY WORDS:** Asyptotic boundary regions; drag forces; oscillating body viscometry.

## 1. INTRODUCTION

Oscillating systems have been used to measure viscosity of fluids with a high degree of accuracy in critical and noncritical fluids [1, 2]. These viscometers consist of an axially symmetric body suspended in a fluid or containing fluid. The viscosity of the fluid causes a torque on the surface of the body which is observed through the decay rate of the amplitude of the angular displacement in a oscillating body and the change in the period of rotation in a rotating body.

In this paper our interest is in cup viscometers. A cup viscometer is a hollow cylinder with the fluid contained inside. The arrangement is shown in Fig. 1 for a partially filled cup. The variable $h$ denotes the height of the liquid in partially filled cup or the internal half-height of a filled cup for which $h = H/2$. The region outside is assumed to be a vacuum. If the

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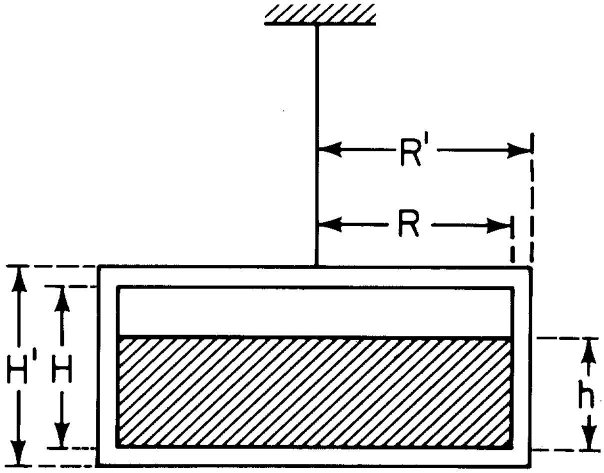


**Fig. 1.** A schematic representation of a partially filled oscillating-cup viscometer.

outside region were to contain a fluid the torque on the outside surfaces can be included by applying the theory of disk viscometers [3–5].

Recently accurate measurements of the viscosity of binary mixtures close to the liquid–liquid critical point or consolute point have been made using an oscillating cup viscometer designed to minimize shear gradient and frequency effects [6]. A further effect that becomes important is due to gravity as the mixture tends to phase separate close to the critical point. A design that minimizes gravity effects is a cup for which the internal height $h$ is made small, $\leqslant 5$ mm, while for stability reasons the radius $R$ is made much larger, typically 2–3 cm. The current working equations for the torque which relate the viscosity of the fluid to the amplitude decrement or period change are valid only for a limited range of values of $h$ (and $R$) with respect to the boundary layer thickness defined by[2]

$$\delta = (\nu/\omega_0)^{1/2} \tag{1}$$

Here $\nu$ is the kinematic viscosity of the fluid and $\omega_0$ is the frequency of oscillation of the cup with the fluid removed. When $h$ becomes small, approaching $\delta$ the current theory for working equations becomes invalid.

A variety of equivalent and exact equations for the torque were constructed by Kestin and Newell [7] but are all in the form of infinite series. Truncated expressions were obtained by Beckwith and Newell [8], but the validity of their results is limited to two regions. These regions, denoted

[2] Definitions of the symbols are given under Nomenclature.

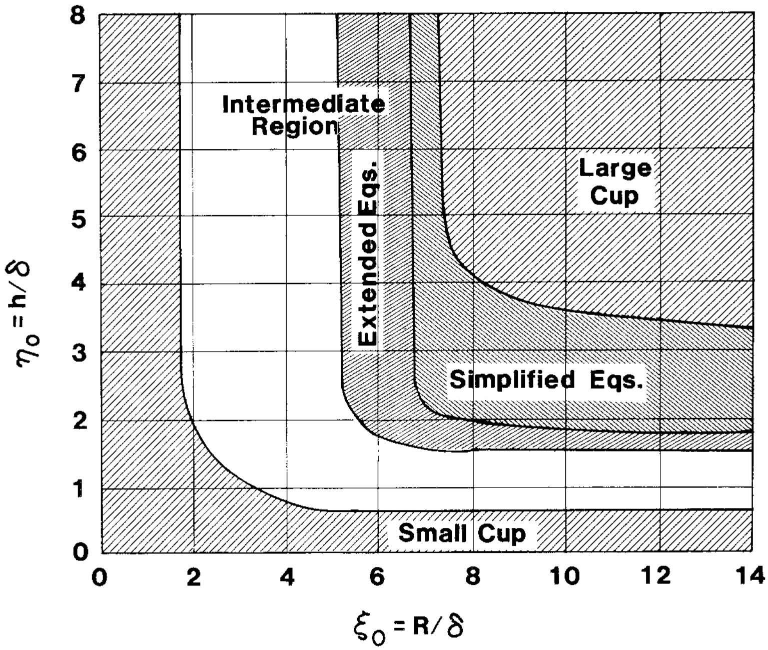


**Fig. 2.** The regions in the $\xi_0$, $\eta_0$ plane where the formulae of Beckwith and Newell [8], the simplified equations with error term (24), and full equations (23) apply.

small and large cup regions, are shown in Fig. 2. The regions of validity are defined to be where the error made in truncating the expression for the torque is less than 1%. If $R$ remains large with respect to $\delta$, the intermediate or disallowed region occurs in the region $h \cong \delta$. In a design which minimizes gravity effects this is exactly the region of interest. For mixtures typically used in consolute point measurements [6] $\delta \cong 0.5$ mm but can approach 1 mm. Consequently any practical design would have $h > \delta$, but possibility $h/\delta < 3.5$, which is the limit of the current theory [9].

The aim of this paper is to obtain closed-form, truncated expressions for the torque, denoted by $D$, in that part of the intermediate region where $h/\delta \cong 1$ and $R/\delta$ is large. Although the system considered is an oscillating cup, expressions obtained for the fluid torque also apply to a rotating cup.

## 2. EXACT EQUATIONS

A complete exposition of the equations of motion of an oscillating cup is given elsewhere [7]; here we give a brief summary. After an initial transient of the order of a period the angular displacement $\alpha$ of the cup as a function of the dimensionless time $\tau = \omega_0 t$ is given by

$$\alpha(\tau) = \alpha(0)\, e^{-\Delta\theta\tau} \cos(\theta\tau + \phi) \tag{2}$$

where $\theta = \omega/\omega_0$ and $\omega$ is the frequency of oscillation with the fluid present. The damping constant $\Delta$ and frequency $\omega$ are related to the roots

$$\zeta = (-\Delta \pm i)\,\theta \tag{3}$$

of the characteristic equation

$$(\zeta + \Delta_0)^2 + 1 + D(\zeta) = 0 \tag{4}$$

$D$ is the Laplace transform of the torque function and $\zeta$ is the transform variable corresponding to $\tau$. Once $D(\zeta)$ is determined in terms of the density and viscosity of the fluid, the decrement and period change are simply given by the real and imaginary parts of a root of Eq. (4).

Kestin and Newell [7] obtained expressions for $D(\zeta)$ by linearizing the Navier–Stokes equations and neglecting secondary flow effects. Apart from these assumptions they obtain by straightforward techniques an expression for $D(\zeta)$ in the form

$$D(\zeta) = \frac{I'\zeta^2}{I}\left(\frac{\tanh(\zeta^{1/2}\eta_0)}{\zeta^{1/2}\eta_0} + \frac{32\zeta}{\pi^2\xi_0}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2\,\zeta_m^3}\,\frac{I_2(\zeta_m\xi_0)}{I_1(\zeta_m\xi_0)}\right) \tag{5}$$

where $\xi_0 = R/\delta$, $\eta_0 = h/\delta$, $I$ is the moment of inertia of the cup and suspension system, and

$$I' = \pi\rho R^4 h \times \begin{cases} 1 & \text{for a filled cup} \\ 1/2 & \text{for a partially filled cup} \end{cases} \tag{6a}$$

is the moment of inertia of the fluid in the cup. $I_1$ and $I_2$ are modified Bessel functions and

$$\zeta_m^2 = \zeta + \left(\frac{(2m+1)\,\pi}{2\eta_0}\right)^2 \tag{6b}$$

## 3. APPROXIMATE EQUATIONS

The expression (5) for $D(\zeta)$ is not useful since it is contains an infinite series. The main contribution of this paper is to partially resum the series to obtain an asymptotic series in powers of $\xi_0^{-1}$ and $\exp(-\eta_0)$. A simple Euler–Maclaurin expansion in powers of $\eta_0^{-1}$ [8] yields only the first-order term and is not useful in the limit as $\eta_0$ approaches 1.

We begin the analysis by noting that $|\zeta| \cong 1$. In most circumstances $I'$ is smaller than $I$ by two decades. Since $\Delta$ is $O(I'/I\xi_0)$ and $\theta = 1 + O(\Delta)$ (see, for example, Ref. 9), it follows that $\zeta \cong i$. When an asymptotic expansion of $I_1(\zeta_m \xi_0)/I_2(\zeta_m \xi_0)$ for large $\xi_0$ is made, the second term in Eq. (5) becomes the sum of terms of the form

$$S_n = \xi_0^{-n} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2 \zeta_m^{n+2}} \tag{7}$$

where we consider $n = 1$ to 4. It is convenient to introduce the identity

$$\zeta_m^{-n} = \zeta^{-n/2} \left(1 + \frac{\zeta^{n/2} - \zeta_m^n}{\zeta_m^n}\right) \tag{8}$$

and use the result that $\sum_{m=0}^{\infty} (2m+1)^{-2} = \pi^2/8$ to write

$$S_n = \xi_0^{-n} \zeta^{-(n+2)/2} (\pi^2/8 + S'_n/2) \tag{9}$$

where

$$S'_n = \sum_{m=-\infty}^{\infty} \frac{\zeta^{(n+2)/2} - \zeta_m^{n+2}}{(2m+1)^2 \zeta_m^{n+2}} \tag{10}$$

With the definition

$$f_n(t) = \frac{\zeta^{(n+2)/2} - [\zeta + (2t+1)^2 x^{-2}]^{(n+2)/2}}{(2t+1)^2 [\zeta + (2t+1)^2 x^{-2}]^{(n+2)/2}} \tag{11}$$

where $x = 2\eta_0/\pi$, it is shown in the Appendix that $S'_n$ can be written as

$$S'_n = \int_{-\infty}^{\infty} f_n(t)\, dt + J_{1,n} + J_{2,n} \tag{12}$$

where

$$J_{1,n} = \frac{i}{2} \zeta^{1/2} x \int_{-\infty}^{(0+)} \frac{f_n(y)}{1 + e^{-\pi x \zeta^{1/2}(y-1)}}\, dy \tag{13}$$

The notation used means that the path of integration begins at $y = -\infty$, encircles the origin once in a positive sense, and returns to the starting point. The variable $t$ is related to $y$ by

$$t = (-1 + ix\zeta^{1/2}(1-y))/2 \tag{14}$$

Note that $f_n(t)$ is analytic in the $t$-plane except for branch points at $t = -\frac{1}{2} \pm \frac{1}{2} ix\zeta^{1/2}$. Had the series not been separated according to Eqs. (9)

and (10), the analyticity of the resulting form for $f_n(t)$ would be more complicated. $J_{2,n}$ is obtained from $J_{1,n}$ letting $i \to -i$ throughout. Straightforward manipulation shows that $J_{2,n} = J_{1,n}$. The integral in Eq. (12) is calculated by using elementary trigonometric transformations.

As an aside we note that an identity similar to Eq. (12) gives the Abel–Plana formula from which the Euler–Maclaurin formula can be obtained [10]. However, the Euler–Maclaurin formula fails to give all but the first term for $S'_n$. The next term in the formula is a power series in $\eta_0^{-1}$ where the coefficients contain higher-order derivatives of $f_n(t)$ at $t = \pm\infty$ which all vanish. This indicates that the next term should be $O[\exp(-\eta_0)]$. In Eq. (12) this term is contained in $J_{1,n}$.

For even $n$ the path of integration in Eq. (13) can be closed and $S_n$ calculated exactly. The expressions are

$$S_2 = \frac{\pi^2}{8\zeta^2\xi_0^2}\left(1 + \frac{1}{2}\operatorname{sech}^2(z/2) - \frac{3}{z}\tanh(z/2)\right) \tag{15a}$$

$$S_4 = \frac{\pi^2}{8\zeta^3\xi_0^4}\left(1 + \frac{7}{8}\operatorname{sech}^2(z/2) - \frac{15}{4z}\tanh(z/2) + \frac{1}{8} z \sinh(z/2)\operatorname{sech}^3(z/2)\right) \tag{15b}$$

where $z = 2\eta_0\zeta^{1/2}$. Asymptotic expansions for large $z$ are

$$S_2 = \frac{\pi^2}{8\zeta^2\xi_0^2}(1 - 3/z) + O(\xi_0^{-2}e^{-z}) \tag{16a}$$

and

$$S_4 = \frac{\pi^2}{8\zeta^3\xi_0^4}(1 - 15/4z) + O(\xi_0^{-4}e^{-z}) \tag{16b}$$

For odd $n$ an asymptotic series in powers of $e^{-x}$ is obtained from $J_{1,n}$ by making use of Watson's lemma for loop integrals [11]. Briefly Watson's lemma states that if

$$f_n(y) \sim \sum_{m=0}^{\infty} a_{n,m}\, y^{m-n/2} \tag{17}$$

as $y \to 0$ and $|ph(y)| \leqslant \pi$, then

$$\frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{zy} f_n(y)\, dy \sim \sum_{m=0}^{\infty} \Gamma(n/2 - m)^{-1}\, a_{n,m} z^{-m+n/2+1} \tag{18}$$

as $z \to \infty$ and $|ph(z)| < \pi/2$. A complete exposition of Watson's lemma is given in Ref. 11. To apply the lemma a large $x$ expansion of the denominator in the integrand of Eq. (13) is made. For $n=1$ the following expansion is obtained

$$J_{1,1} = -\frac{1}{2}(2\pi^3/z)^{1/2} e^{-z} \sum_{m=0}^{\infty} a_{1,m} z^{-m} \frac{\Gamma(3/2)}{\Gamma(3/2-m)} + (\pi^3/z)^{1/2} e^{-2z} \times \sum_{m=0}^{\infty} a_{1,m}(2z)^{-m} \frac{\Gamma(3/2)}{\Gamma(3/2-m)} + O(e^{-3z}/z^{1/2}) \tag{19}$$

where, again, $z = \zeta^{1/2}\eta_0$. The first four coefficients are

$$a_{1,0} = 1, \qquad a_{1,1} = 11/8, \qquad a_{1,2} = -159/128, \qquad a_{1,3} = 2865/1024$$

For $n=3$ we obtain

$$J_{3,1} = -\frac{2}{3}(\pi^3 z/2)^{1/2} e^{-z} \sum_{m=0}^{\infty} a_{3,m} z^{-m} \frac{\Gamma(5/2)}{\Gamma(5/2-m)} + O(z^{1/2} e^{-2z}) \tag{20}$$

where $a_{3,0} = 1$ and $a_{3,1} = 13/4$.

In conclusion, the final results are

$$S_1 = \zeta^{-3/2}\xi_0^{-1}\left(\frac{\pi^2}{8} - \frac{\pi}{2\zeta^{1/2}\eta_0} + J_{1,1}\right) \tag{21}$$

$$S_3 = \zeta^{-5/2}\xi_0^{-3}\left(\frac{\pi^2}{8} - \frac{2\pi}{3\zeta^{1/2}\eta_0} + J_{3,1}\right) \tag{22}$$

## 4. SUMMARY

The torque $D(\zeta)$ on the interior of the cup is given by

$$D(\zeta) = \frac{I'\zeta^2}{I}\left(\frac{\tanh(\zeta^{1/2}\eta_0)}{\zeta^{1/2}\eta_0} + \frac{32\zeta}{\pi^2}(S_1 - 3S_2/2 + 3S_3/8 + 3S_4/8)\right) \tag{23}$$

where $S_n$ are given by Eqs. (15), (21), and (22). To illustrate this result for a cup design for which $\xi_0 \geqslant 6$, we simplify Eq. (23) and retain in $S_1$ only the first series for $J_{3,1}$, neglect $J_{3,1}$ from $S_3$, and neglect $S_4$. The error $E(\zeta)$ incurred in $D(\zeta)$ is then

$$E(\zeta) \leqslant \frac{I'\zeta^2}{I}\left(32\xi_0^{-1}\zeta^{-1/2}(\pi z)^{-1/2} e^{-2z} + 2\xi_0^{-3}\zeta^{-3/2}(2z/\pi)^{1/2} e^{-z} + \frac{3}{2\xi_0^4\zeta^2}\right) \tag{24}$$

In Fig. 2 the region for which $E(\zeta)$ is 0.1% of $D(\zeta)$ is denoted the simplified region. For comparison the results obtained by Beckwith and Newell [8], denoted small and large cup are reproduced. Note that even with the simplifications the large cup region has been substantially extended into the intermetiate region. The region of validity the full equations is also shown.

As an aside we note that the large cup region is now also extended into the region of smaller values of $\xi_0$. This is due mainly to retaining the term of $O(\xi_0^{-4})$ in the full equations. Recent viscosity experiments [12] using viscometer designs for which $\eta_0 \simeq 4$–5 but $\xi_0 \simeq 5$ will thus also benefit from the above results.

## APPENDIX

The series expression (10) is transformed to integral form by making use of the well-known formula [13],

$$\sum_{j=-N}^{N} f(j) = (2i)^{-1} \int_{\mathscr{C}} \cot(\pi t)\, dt \tag{25}$$

where $\mathscr{C}$ is the closed contour shown in Fig. 3. The contour crosses the real-$t$ axis at $N+\delta$ and $-N-\delta$, where $0<\delta<1$. We denote by $\mathscr{C}_1$ and $\mathscr{C}_2$

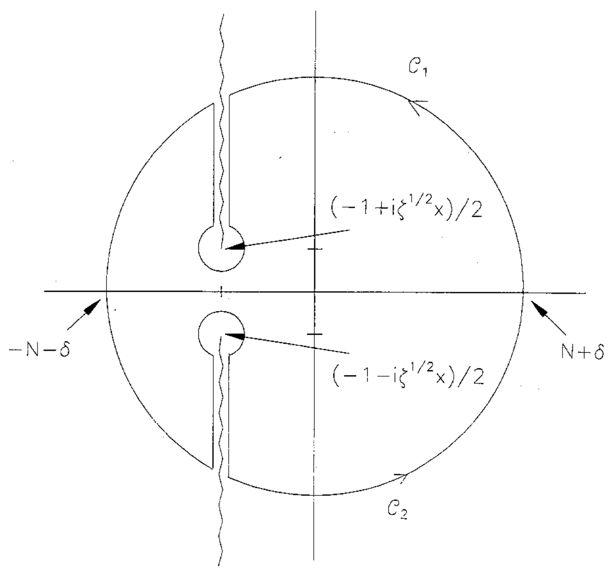


**Fig. 3.** The $t$-plane contour $\mathscr{C}$ for the integral in Eq. (25).

the upper and lower parts of $\mathscr{C}$, then since $f(t)$ is analytic within the contour, by Cauchy's theorem,

$$\int_{-N-\delta}^{N+\delta} f(t)\,dt + \int_{\mathscr{C}_1} f(t)\,dt = 0 \tag{26}$$

and

$$\int_{-N-\delta}^{N+\delta} f(t)\,dt - \int_{\mathscr{C}_2} f(t)\,dt = 0 \tag{27}$$

With these expressions, Eq. (25) can be written as

$$\begin{aligned}\sum_{j=-N}^{N} f(j) - \int_{-N-\delta}^{N+\delta} f(t)\,dt &= (2i)^{-1} \int_{\mathscr{C}_1+\mathscr{C}_2} \cot(\pi t)\, f(t)\,dt \\ &\quad + \frac{1}{2}\int_{\mathscr{C}_1} f(t)\,dt - \frac{1}{2}\int_{\mathscr{C}_2} f(t)\,dt \\ &= \int_{\mathscr{C}_1} f(t)(1-e^{-2\pi i t})\,dt + \int_{\mathscr{C}_2} f(t)(e^{2\pi i t}-1)\,dt\end{aligned}$$

In the limit that $N \to \infty$, the integrand vanishes along the arcs of $\mathscr{C}_1$ and $\mathscr{C}_2$ and the only remaining terms are the integrals along the branch cuts. These integrals are transformed to the form given in Eq. (13) by making the transformation (14).

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## NOMENCLATURE

| | |
|---|---|
| $D(\zeta)$ | Torque on the cup, Eq. (5) |
| $E(\zeta)$ | Truncation error term, Eq. (24) |
| $h$ | Internal half-height of a filled cup or the height of the liquid in a partially filled cup |

| | |
|---|---|
| $I$ | Moment of inertia of cup and suspension system |
| $I'$ | Moment of inertia of fluid inside cup |
| $I_n$ | Modified Bessel function of order $n$ |
| $J_{i,n}$ | Defined in Eq. (13) |
| $R$ | Radius of the cup |
| $S_n$ | Defined in Eq. (7) |
| $S'_n$ | Defined in Eq. (10) |
| $x$ | Variable $2\eta_0/\pi$ |
| $z$ | Variable $2\eta_0\zeta^{1/2}$ |
| $\alpha(\tau)$ | Angular displacement of the cup |
| $\delta$ | Boundary layer thickness |
| $\Delta$ | Logrithmic decrement |
| $\zeta$ | Laplace transform variable |
| $\eta_0$ | Dimensionless height $h/\delta$ |
| $\theta$ | Frequency ratio $\omega/\omega_0$ |
| $\nu$ | Kinematic viscosity |
| $\xi_0$ | Dimensionless radius $R/\delta$ |
| $\rho$ | Density of liquid |
| $\tau$ | Dimensionless time $\omega_0 t$ |
| $\phi$ | Phase angle of oscillation |
| $\omega$ | Angular frequency of oscillation with liquid present in cup |
| $\omega_0$ | Angular frequency of oscillation in a vacuum |

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